

# Boundary value problems for the diffusion equation of the variable order in differential and difference settings

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## Abstract

Solutions of boundary value problems for a diffusion equation of fractional and variable order in differential and difference settings are studied. It is shown that the method of the energy inequalities is applicable to obtaining a priori estimates for these problems exactly as in the classical case. The credibility of the obtained results is verified by performing numerical calculations for a test problem.

*Keywords:* fractional derivative, a priori estimate, difference scheme, stability and convergence

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## 1. Introduction

Fractional calculus is used for the description of a large class of physical and chemical processes that occur in media with fractal geometry as well as in the mathematical modeling of economic and social-biological phenomena [1, 2, 3, 4, 5]. In general, a medium in which a process proceeds is not homogenous, moreover, its properties may vary in time. Mathematical models containing equations with variable order derivatives provide a more accurate and realistic description of processes proceeding in such complex media (see e.g. [6, 7, 8]).

Therefore, the development of numerical and analytical methods of the theory of fractional order differential equations is an actual and important problem.

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Numerical methods for solving variable order fractional differential equations with various kinds of the variable order fractional derivative have been proposed [9, 10, 11, 12, 13, 14].

The positivity of the fractional derivative operator has been proved in [1] and this result allows to obtain a priori estimates for solutions of a large class of boundary value problems for the equations containing fractional derivatives. The authors of the paper [15] have obtained a priori estimate for the solution of the Dirichlet boundary value problem of a fractional order diffusion equation in terms of a fractional Riemann–Liouville integral. The fractional diffusion equation with the regularized fractional derivative has been studied, for example, in [16]. In the papers [17, 18, 19], the diffusion-wave equation with Caputo and Riemann–Liouville fractional derivatives has been studied. The difference schemes for boundary value problems for the fractional diffusion equation both in one and multidimensional cases have been studied [15, 20, 21]. A priori estimates for the difference problems obtained in [15, 20, 21] by using the maximum principle imply the stability and convergence of the considered difference schemes.

Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the diffusion-wave equation with Caputo fractional derivative have been obtained [22]. More references on fractional order differential equations, including the diffusion-wave equation, can be found, for example, in [23].

## 2. Boundary value problems in differential setting

### 2.1. The Dirichlet boundary value problem

In rectangle  $\bar{Q}_T = \{(x, t) : 0 \leq x \leq l, 0 \leq t \leq T\}$  let us study the boundary value problem

$$\partial_{0t}^{\alpha(x)} u = \frac{\partial}{\partial x} \left( k(x, t) \frac{\partial u}{\partial x} \right) - q(x, t)u + f(x, t), \quad 0 < x < l, 0 < t \leq T, \quad (1)$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \leq t \leq T, \quad (2)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq l. \quad (3)$$

Where  $0 < c_1 \leq k(x, t) \leq c_2$ ,  $q(x, t) \geq 0$ ,  $\partial_{0t}^{\alpha(x)} u(x, t) = \int_0^t u_\tau(x, \tau)(t - \tau)^{-\alpha(x)} d\tau / \Gamma(1 - \alpha(x))$  is a Caputo fractional derivative of order  $\alpha(x)$ ,  $0 < \alpha(x) < 1$ , for all  $x \in (0, l)$ ,  $\alpha(x) \in C(0, T)$  [18, 24].

Suppose further the existence of a solution  $u(x, t) \in C^{2,1}(\bar{Q}_T)$  for the problem (1)–(3), where  $C^{m,n}$  is the class of functions, continuous together with their partial derivatives of the order  $m$  with respect to  $x$  and order  $n$  with respect to  $t$  on  $\bar{Q}_T$ .

The existence of the solution for the initial boundary value problem of a number of fractional order differential equation has been proved [25, 26, 27].

Let us prove the following:

**Lemma 1.** For any functions  $v(t)$  and  $w(t)$  absolutely continuous on  $[0, T]$ , one has the equality:

$$\begin{aligned} v(t)\partial_{0t}^\beta w(t) + w(t)\partial_{0t}^\beta v(t) &= \partial_{0t}^\beta (v(t)w(t)) + \\ &+ \frac{\beta}{\Gamma(1-\beta)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\beta}} \int_0^\xi \frac{v'(\eta)d\eta}{(t-\eta)^\beta} \int_0^\xi \frac{w'(s)ds}{(t-s)^\beta}, \end{aligned} \quad (4)$$

where  $0 < \beta < 1$ .

**Proof.** Let us consider the difference

$$\begin{aligned} v(t)\partial_{0t}^\beta w(t) + w(t)\partial_{0t}^\beta v(t) - \partial_{0t}^\beta (v(t)w(t)) &= \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{w'(s)(v(t) - v(s)) + v'(s)(w(t) - w(s))}{(t-s)^\beta} ds = \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{v'(s)ds}{(t-s)^\beta} \int_s^t w'(\xi)d\xi + \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{w'(s)ds}{(t-s)^\beta} \int_s^t v'(\xi)d\xi = \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t w'(\xi)d\xi \int_0^\xi \frac{v'(s)ds}{(t-s)^\beta} + \frac{1}{\Gamma(1-\beta)} \int_0^t v'(\xi)d\xi \int_0^\xi \frac{w'(s)ds}{(t-s)^\beta} = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\xi)^\beta \frac{\partial}{\partial \xi} \left( \int_0^\xi \frac{v'(\eta) d\eta}{(t-\eta)^\beta} \int_0^\xi \frac{w'(s) ds}{(t-s)^\beta} \right) d\xi = \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\beta}} \int_0^\xi \frac{v'(\eta) d\eta}{(t-\eta)^\beta} \int_0^\xi \frac{w'(s) ds}{(t-s)^\beta}.
\end{aligned}$$

The proof of the lemma is complete.

If  $v(t) = w(t)$  then from the Lemma 1 one has the following:

**Corollary 1.** For any function  $v(t)$  absolutely continuous on  $[0, T]$ , the following equality takes place:

$$v(t) \partial_{0t}^\beta v(t) = \frac{1}{2} \partial_{0t}^\beta v^2(t) + \frac{\beta}{2\Gamma(1-\beta)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\beta}} \left( \int_0^\xi \frac{v'(\eta) d\eta}{(t-\eta)^\beta} \right)^2, \quad (5)$$

where  $0 < \beta < 1$ .

Let us use the following notation:  $\|u\|_0^2 = \int_0^l u^2(x, t) dx$ ,  $D_{0t}^{-\beta} u(x, t) = \int_0^t (t-s)^{\beta-1} u(x, s) ds / \Gamma(\beta)$  – fractional Riemann–Liouville integral of order  $\beta$ .

**Theorem 1.** If  $k(x, t) \in C^{1,0}(\bar{Q}_T)$ ,  $q(x, t), f(x, t) \in C(\bar{Q}_T)$ ,  $k(x, t) \geq c_1 > 0$ ,  $q(x, t) \geq 0$  everywhere on  $\bar{Q}_T$ , then the solution  $u(x, t)$  of the problem (1)–(3) satisfies the a priori estimate:

$$\begin{aligned}
&\int_0^l D_{0t}^{\alpha(x)-1} u^2(x, t) dx + c_1 \int_0^t \|u_x(x, s)\|_0^2 ds \leq \\
&\leq \frac{l^2}{2c_1} \int_0^t \|f(x, s)\|_0^2 ds + \int_0^l \frac{t^{1-\alpha(x)}}{\Gamma(2-\alpha(x))} u_0^2(x) dx.
\end{aligned} \quad (6)$$

**Proof.** Let us multiply equation (1) by  $u(x, t)$  and integrate the resulting relation over  $x$  from 0 to  $l$ :

$$\int_0^l u(x, t) \partial_{0t}^{\alpha(x)} u(x, t) dx - \int_0^l u(x, t) (k(x, t) u_x(x, t))_x dx +$$

$$+ \int_0^l q(x, t) u^2(x, t) dx = \int_0^l u(x, t) f(x, t) dx. \quad (7)$$

Then transform the terms in identity (7) as:

$$- \int_0^l u(x, t) (k(x, t) u_x(x, t))_x dx = \int_0^l k(x, t) u_x^2(x, t) dx \geq c_1 \|u_x(x, t)\|_0^2,$$

$$\left| \int_0^l u(x, t) f(x, t) dx \right| \leq \varepsilon \|u(x, t)\|_0^2 + \frac{1}{4\varepsilon} \|f(x, t)\|_0^2, \quad \varepsilon > 0,$$

Using the equality (5) one obtains

$$\int_0^l u(x, t) \partial_{0t}^{\alpha(x)} u(x, t) dx \geq \frac{1}{2} \int_0^l \partial_{0t}^{\alpha(x)} u^2(x, t) dx.$$

Taking into account the above performed transformations, from the identity (7) one arrives at the inequality

$$\frac{1}{2} \int_0^l \partial_{0t}^{\alpha(x)} u^2(x, t) dx + c_1 \|u_x(x, t)\|_0^2 \leq \varepsilon \|u(x, t)\|_0^2 + \frac{1}{4\varepsilon} \|f(x, t)\|_0^2. \quad (8)$$

Using the inequality  $\|u(x, t)\|_0^2 \leq (l^2/2) \|u_x(x, t)\|_0^2$ , from the inequality (8) at  $\varepsilon = c_1/l^2$  one obtains

$$\int_0^l \partial_{0t}^{\alpha(x)} u^2(x, t) dx + c_1 \|u_x(x, t)\|_0^2 \leq \frac{l^2}{2c_1} \|f(x, t)\|_0^2. \quad (9)$$

Changing the variable  $t$  by  $s$  in the inequality (9) and integrating it over  $s$  from 0 to  $t$ , one obtains a priori estimate (6). The uniqueness and the continuous dependence of the solution of the problem (1)–(3) on the input data follow from the a priori estimate (6).

The solution of the problem (1)–(3) with  $\alpha(x) = \alpha$  ( $\alpha = \text{const}$ ) satisfies the a priori estimates:

$$D_{0t}^{\alpha-1} \|u(x, t)\|_0^2 + c_1 \int_0^t \|u_x(x, s)\|_0^2 ds \leq \frac{l^2}{2c_1} \int_0^t \|f(x, s)\|_0^2 ds + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|u_0(x)\|_0^2, \quad (10)$$

$$\|u(x, t)\|_0^2 + D_{0t}^{-\alpha} \|u_x(x, t)\|_0^2 \leq M (D_{0t}^{-\alpha} \|f(x, t)\|_0^2 + \|u_0^2(x)\|_0^2). \quad (11)$$

Inequality (10) follows from (6), and the a priori estimate (11) follows from inequality (8) with  $\alpha(x) = \alpha$ . Actually, applying the fractional integration operator  $D_{0t}^{-\alpha}$  to the both sides of inequality (8), one arrives at the estimate (11), which contains the constant  $M = \max\{l^2/c_1, 1\} / \min\{1, c_1\}$ .

## 2.2. The Robin boundary value problem.

In the problem (1)–(3) we replace the boundary conditions (2) with

$$\begin{cases} k(0, t)u_x(0, t) = \beta_1(t)u(0, t) - \mu_1(t), \\ -k(l, t)u_x(l, t) = \beta_2(t)u(l, t) - \mu_2(t). \end{cases} \quad (12)$$

In the rectangle  $\bar{Q}_T$  we consider the Robin boundary value problem (1), (3), (12).

**Theorem 2.** If  $k(x, t) \in C^{1,0}(\bar{Q}_T)$ ,  $q(x, t), f(x, t) \in C(\bar{Q}_T)$ ,  $k(x, t) \geq c_1 > 0$ ,  $q(x, t) \geq 0$  everywhere on  $\bar{Q}_T$ ,  $\beta_i(t), \mu_i(t) \in C[0, T]$ ,  $\beta_i(t) \geq \beta_0 > 0$ , for all  $t \in [0, T]$ ,  $i = 1, 2$ , then the solution  $u(x, t)$  of the problem (1), (3), (12) satisfies the a priori estimate:

$$\begin{aligned} & \int_0^l D_{0t}^{\alpha(x)-1} u^2(x, t) dx + \gamma \left( \int_0^t (\|u_x(x, s)\|_0^2 + u^2(0, s) + u^2(l, s)) ds \right) \leq \\ & \leq \frac{\delta}{\gamma} \left( \int_0^t (\|f(x, s)\|_0^2 + \mu_1^2(s) + \mu_2^2(s)) ds \right) + \int_0^l \frac{t^{1-\alpha(x)}}{\Gamma(2-\alpha(x))} u_0^2(x) dx, \quad (13) \end{aligned}$$

where  $\gamma = \min\{c_1, \beta_0\}$ ,  $\delta = \max\{1 + l, l^2\}$ .

**Proof.** Multiply the equation (1) by  $u(x, t)$  and integrate the resulting relation over  $x$  from 0 to  $l$ :

$$\int_0^l u \partial_{0t}^\alpha u dx - \int_0^l u (ku_x)_x dx + \int_0^l qu^2 dx = \int_0^l u f dx. \quad (14)$$

Then, transform the terms of the identity (14):

$$\begin{aligned} \int_0^l u(x, t) \partial_{0t}^{\alpha(x)} u(x, t) dx &\geq \frac{1}{2} \int_0^l \partial_{0t}^{\alpha(x)} u^2(x, t) dx. \\ - \int_0^l u (ku_x)_x dx &= \beta_1(t) u^2(0, t) + \beta_2(t) u^2(l, t) - \mu_1(t) u(0, t) - \mu_2(t) u(l, t) + \int_0^l k u_x^2 dx, \end{aligned}$$

$$\left| \int_0^l u f dx \right| \leq \varepsilon \|u\|_0^2 + \frac{1}{4\varepsilon} \|f\|_0^2, \quad \varepsilon > 0.$$

From (14), taking into account the transformations performed, one arrives at the inequality

$$\begin{aligned} \frac{1}{2} \int_0^l \partial_{0t}^{\alpha(x)} u^2(x, t) dx + c_1 \|u_x(x, t)\|_0^2 + \beta_0 u^2(0, t) + \beta_0 u^2(l, t) &\leq \\ &\leq \mu_1(t) u(0, t) + \mu_2(t) u(l, t) + \varepsilon \|u\|_0^2 + \frac{1}{4\varepsilon} \|f\|_0^2. \end{aligned} \quad (15)$$

Using the inequalities  $\mu_1(t) u(0, t) \leq \varepsilon u^2(0, t) + (4\varepsilon)^{-1} \mu_1^2(t)$ ,  $\mu_2(t) u(l, t) \leq \varepsilon u^2(l, t) + (4\varepsilon)^{-1} \mu_2^2(t)$ ,  $\varepsilon > 0$ ;  $\|u(x, t)\|_0^2 \leq l^2 \|u_x(x, t)\|_0^2 + l(u^2(0, t) + u^2(l, t))$  with  $\varepsilon = \gamma/(2\delta)$ , from (15) one has the following inequality

$$\begin{aligned} \int_0^l \partial_{0t}^{\alpha(x)} u^2(x, t) dx + \gamma (\|u_x(x, t)\|_0^2 + u^2(0, t) + u^2(l, t)) &\leq \\ &\leq \frac{\delta}{\gamma} (\|f(x, t)\|_0^2 + \mu_1^2(t) + \mu_2^2(t)). \end{aligned} \quad (16)$$

Changing variable  $t$  by  $s$  in inequality (16) and integrating it over  $s$  from 0 to  $t$ , one obtains the a priori estimate (13).

The uniqueness and the continuous dependence of the solution of problem (1), (3), (12) on the input data follow from the a priori estimate (13).

The solution of the problem (1), (3), (12) with  $\alpha(x) = \alpha$  ( $\alpha = \text{const}$ ) satisfies the following a priori estimates:

$$\begin{aligned} & D_{0t}^{\alpha-1} \|u(x, t)\|_0^2 + \gamma \left( \int_0^t (\|u_x(x, s)\|_0^2 + u^2(0, s) + u^2(l, s)) ds \right) \leq \\ & \leq \frac{\delta}{\gamma} \left( \int_0^t (\|f(x, s)\|_0^2 + \mu_1^2(s) + \mu_2^2(s)) ds \right) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|u_0(x)\|_0^2, \end{aligned} \quad (17)$$

$$\begin{aligned} & \|u(x, t)\|_0^2 + D_{0t}^{-\alpha} \|u_x(x, t)\|_0^2 \leq M(D_{0t}^{-\alpha} \|f(x, t)\|_0^2 + \\ & + D_{0t}^{-\alpha} \mu_1^2(t) + D_{0t}^{-\alpha} \mu_2^2(t) + \|u_0^2(x)\|_0^2), \end{aligned} \quad (18)$$

Inequality (17) follows from (13), and the a priori estimate (18) follows from inequality (16) with  $\alpha(x) = \alpha$ . Actually, applying the fractional integration operator  $D_{0t}^{-\alpha}$  to the both sides of inequality (16), one arrives at the estimate (18), which contains the constant  $M = \max\{\delta/\gamma, 1\}/\min\{1, \gamma\}$ .

### 3. Boundary value problems in difference setting

#### 3.1. The Dirichlet boundary value problem

Suppose that a solution  $u(x, t) \in C^{4,3}(Q_T)$  of the problem (1)–(3) exists, and the coefficients of the equation (1) and the functions  $f(x, t)$ ,  $u_0(x)$  satisfy the smoothness conditions, required for the construction of difference schemes with the order of approximation  $O(\tau + h^2)$ .

In the rectangle  $\bar{Q}_T$  we introduce the grid  $\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$ , where  $\bar{\omega}_h = \{x_i = ih, i = 0, 1, \dots, N, hN = l\}$ ,  $\bar{\omega}_\tau = \{t_j = j\tau, j = 0, 1, \dots, j_0, \tau j_0 = T\}$ .

To problem (1)–(3), we assign the difference scheme:

$$\Delta_{0t_j}^{\alpha_i} y = \Lambda(\sigma y^{j+1} + (1-\sigma)y^j) + \varphi, \quad i = 1, \dots, N-1, \quad j = 1, \dots, j_0-1, \quad (19)$$

$$y(0, t) = 0, \quad y(l, t) = 0, \quad j = 0, \dots, j_0, \quad (20)$$



$$y(x, 0) = u_0(x), \quad i = 0, \dots, N, \quad (21)$$

where  $\Lambda y = (ay_{\bar{x}})_x - dy$ ,  $a = k(x_{i-1/2}, \bar{t})$ ,  $d = q(x_i, \bar{t})$ ,  $\varphi = f(x_i, \bar{t})$ ,  $\bar{t} = t_{j+1/2}$ ,  $0 \leq \sigma \leq 1$ ,  $\Delta_{0t_j}^{\alpha_i} y = \sum_{s=0}^j (t_{j-s+1}^{1-\alpha_i} - t_{j-s}^{1-\alpha_i}) y_t^s / \Gamma(2 - \alpha_i)$  – a difference analogue of the Caputo fractional derivative of order  $\alpha_i$ ,  $\alpha_i = \alpha(x_i)$  [15].

According to [15, 28] the difference scheme (19)–(21) has the order of approximation  $O(\tau + h^2)$ .

**Lemma 2.** For every function  $y(t)$  defined on the grid  $\bar{\omega}_\tau$  one has the inequalities

$$y^{j+1} \Delta_{0t}^\alpha y \geq \frac{1}{2} \Delta_{0t}^\alpha (y^2) + \frac{\tau^\alpha \Gamma(2 - \alpha)}{2} (\Delta_{0t}^\alpha y)^2, \quad (22)$$

$$y^j \Delta_{0t}^\alpha y \geq \frac{1}{2} \Delta_{0t}^\alpha (y^2) - \frac{\tau^\alpha \Gamma(2 - \alpha)}{2(2 - 2^{1-\alpha})} (\Delta_{0t}^\alpha y)^2. \quad (23)$$

**Proof.** Inequality (22) is equivalent to the inequality

$$\begin{aligned} y^{j+1} \Delta_{0t}^\alpha y - \frac{1}{2} \Delta_{0t}^\alpha (y^2) - \frac{\tau^\alpha \Gamma(2 - \alpha)}{2} (\Delta_{0t}^\alpha y)^2 &= \frac{1}{\Gamma(2 - \alpha)} y^{j+1} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s - \\ &\quad - \frac{1}{\Gamma(2 - \alpha)} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s \frac{y^{s+1} + y^s}{2} - \frac{\tau^\alpha \Gamma(2 - \alpha)}{2} (\Delta_{0t}^\alpha y)^2 = \\ &= \frac{1}{\Gamma(2 - \alpha)} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s (y^{j+1} - \frac{y^{s+1} + y^s}{2}) - \frac{\tau^\alpha \Gamma(2 - \alpha)}{2} (\Delta_{0t}^\alpha y)^2 = \\ &= \frac{1}{\Gamma(2 - \alpha)} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s (\frac{\tau}{2} y_t^s + \sum_{k=s+1}^j y_t^k \tau) - \frac{\tau^\alpha \Gamma(2 - \alpha)}{2} (\Delta_{0t}^\alpha y)^2 = \\ &= \frac{\tau}{2\Gamma(2 - \alpha)} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) (y_t^s)^2 + \frac{1}{\Gamma(2 - \alpha)} \sum_{k=1}^j y_t^k \tau \sum_{s=0}^{k-1} (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s - \\ &\quad - \frac{\tau^\alpha \Gamma(2 - \alpha)}{2} (\Delta_{0t}^\alpha y)^2 \geq 0. \end{aligned} \quad (24)$$

Here we consider the sums to be equal to zero if the upper summation limit is less than the lower one.

Let us introduce the following notation:  $\sum_{s=0}^k (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s = v^{k+1}$ , then  $y_t^0 = (t_{j+1}^{1-\alpha} - t_j^{1-\alpha})^{-1} v^1$ ,  $y_t^k = \tau(t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} v_t^k$ ,  $k = 1, 2, \dots, j$ . Taking into account the introduced notation, we rewrite the inequality (24) as

$$\begin{aligned}
& \frac{\tau}{2\Gamma(2-\alpha)} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha})^{-1} (v^1)^2 + \frac{\tau}{2\Gamma(2-\alpha)} \sum_{k=1}^j \tau^2 (t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} (v_t^k)^2 + \\
& + \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^j \tau^2 (t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} v_t^k v^k - \frac{\tau^\alpha}{2\Gamma(2-\alpha)} (v^{j+1})^2 = \\
& = \frac{\tau}{2\Gamma(2-\alpha)} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha})^{-1} (v^1)^2 + \frac{1}{2\Gamma(2-\alpha)} \sum_{k=1}^j \tau (t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} ((v^{k+1})^2 - (v^k)^2) - \\
& - \frac{\tau^\alpha}{2\Gamma(2-\alpha)} (v^{j+1})^2 = \frac{1}{2\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \tau ((t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} - \\
& - (t_{j-k}^{1-\alpha} - t_{j-k-1}^{1-\alpha})^{-1}) (v^{k+1})^2 \geq 0. \tag{25}
\end{aligned}$$

Obviously, inequality (25) is valid since  $(t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} - (t_{j-k}^{1-\alpha} - t_{j-k-1}^{1-\alpha})^{-1} > 0$ ,  $k = 0, 1, \dots, j-1$ .

Let us prove now the inequality (23). Since  $y^j = y^{j+1} - \tau y_t$ , one obtains

$$\begin{aligned}
& y^j \Delta_{0t}^\alpha y - \frac{1}{2} \Delta_{0t}^\alpha (y^2) + \frac{\tau^\alpha \Gamma(2-\alpha)}{2(2-2^{1-\alpha})} (\Delta_{0t}^\alpha y)^2 = \\
& = y^{j+1} \Delta_{0t}^\alpha y - \frac{1}{2} \Delta_{0t}^\alpha (y^2) + \frac{\tau^\alpha \Gamma(2-\alpha)}{2(2-2^{1-\alpha})} (\Delta_{0t}^\alpha y)^2 - \tau y_t \Delta_{0t}^\alpha y = \\
& = \frac{\tau^\alpha (3-2^{1-\alpha})}{2\Gamma(2-\alpha)(2-2^{1-\alpha})} (v^{j+1})^2 - \frac{\tau^{1+\alpha}}{\Gamma(2-\alpha)} v_t^j v^{j+1} + \frac{1}{2\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \tau ((t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} - \\
& - (t_{j-k}^{1-\alpha} - t_{j-k-1}^{1-\alpha})^{-1}) (v^{k+1})^2 = \frac{\tau^\alpha (2^{1-\alpha} - 1)}{2\Gamma(2-\alpha)(2-2^{1-\alpha})} (v^{j+1})^2 + \frac{\tau^\alpha}{\Gamma(2-\alpha)} v^{j+1} v^j + \\
& + \frac{\tau^\alpha (2-2^{1-\alpha})}{2\Gamma(2-\alpha)(2^{1-\alpha}-1)} (v^j)^2 + \frac{1}{2\Gamma(2-\alpha)} \sum_{k=0}^{j-2} \tau ((t_{j-k+1}^{1-\alpha} - t_{j-k}^{1-\alpha})^{-1} - \\
& - (t_{j-k}^{1-\alpha} - t_{j-k-1}^{1-\alpha})^{-1}) (v^{k+1})^2 \geq 0.
\end{aligned}$$

The proof of the lemma is complete.

**Corollary 2.** For any function  $y(t)$  defined on the grid  $\bar{\omega}_\tau$  one has the inequality

$$(\sigma y^{j+1} + (1-\sigma)y^j) \Delta_{0t}^\alpha y \geq \frac{1}{2} \Delta_{0t}^\alpha (y^2) + \frac{\tau^\alpha \Gamma(2-\alpha)}{2(2-2^{1-\alpha})} ((3-2^{1-\alpha})\sigma - 1) (\Delta_{0t}^\alpha y)^2. \quad (26)$$

**Theorem 3.** The difference scheme (19)–(21) at  $\sigma \geq 1/(3-2^{1-\min(\alpha_i)})$  is absolutely stable and its solution satisfies the following a priori estimate:

$$\begin{aligned} & \left( \frac{1}{\Gamma(2-\alpha_i)}, \sum_{j'=0}^j (t_{j-j'+1}^{1-\alpha_i} - t_{j-j'}^{1-\alpha_i}) (y^{j'+1})^2 \right) + \\ & + c_1 \sum_{j'=0}^j \|\sigma y_{\bar{x}}^{j'+1} + (1-\sigma)y_{\bar{x}}^{j'}\|_0^2 \tau \leq \\ & \leq \frac{l^2}{2c_1} \sum_{j'=0}^j \|\varphi^{j'}\|_0^2 \tau + \left( \frac{t_{j+1}^{1-\alpha_i}}{\Gamma(2-\alpha_i)}, u_0^2(x_i) \right), \end{aligned} \quad (27)$$

where  $(y, v) = \sum_{i=1}^{N-1} y_i v_i h$ ,  $(y, v] = \sum_{i=1}^N y_i v_i h$ ,  $\|y\|_0^2 = (y, y)$ ,  $\|y\|_0^2 = (y, y]$ .

**Proof.** Let us multiply scalarly equation (19) by  $y^{(\sigma)} = \sigma y^{j+1} + (1-\sigma)y^j$ :

$$(\Delta_{0t_j}^{\alpha_i} y, y^{(\sigma)}) - (\Lambda y^{(\sigma)}, y^{(\sigma)}) = (\varphi, y^{(\sigma)}). \quad (28)$$

Let us transform the terms in identity (28):

$$\begin{aligned} & (y^{(\sigma)}, \Delta_{0t_j}^{\alpha_i} y) \geq \frac{1}{2} (1, \Delta_{0t_j}^{\alpha_i} (y^2)) + \\ & + \left( \frac{\tau^{\alpha_i} \Gamma(2-\alpha_i)}{2(2-2^{1-\alpha_i})} ((3-2^{1-\alpha_i})\sigma - 1), (\Delta_{0t_j}^{\alpha_i} y)^2 \right), \\ & -(\Lambda y^{(\sigma)}, y^{(\sigma)}) = (a, (y_{\bar{x}}^{(\sigma)})^2] + (d, (y^{(\sigma)})^2), \\ & |(\varphi, y^{(\sigma)})| \leq \varepsilon \|y^{(\sigma)}\|_0^2 + \frac{1}{4\varepsilon} \|\varphi\|_0^2, \quad \varepsilon > 0. \end{aligned}$$

Taking into account the above-performed transformations, from identity (28) at  $\sigma \geq 1/(3-2^{1-\min(\alpha_i)})$  one arrives at the inequality

$$\frac{1}{2} (1, \Delta_{0t_j}^{\alpha_i} (y^2)) + c_1 \|y_{\bar{x}}^{(\sigma)}\|_0^2 \leq \varepsilon \|y^{(\sigma)}\|_0^2 + \frac{1}{4\varepsilon} \|\varphi\|_0^2. \quad (29)$$

From (29) at  $\varepsilon = c_1/l^2$ , using that  $\|y\|_0^2 \leq (l^2/2)\|y_{\bar{x}}\|_0^2$ , one obtains the inequality

$$(1, \Delta_{0t_j}^{\alpha_i}(y^2)) + c_1\|y_{\bar{x}}^{(\sigma)}\|_0^2 \leq \frac{l^2}{2c_1}\|\varphi\|_0^2. \quad (30)$$

Multiplying the inequality (30) by  $\tau$  and summing over  $j'$  from 0 to  $j$ , one obtains the a priori estimate (27). The stability and convergence of the difference scheme (19)–(21) follow from the a priori estimate (27).

If  $\alpha(x) = \alpha$  ( $\alpha = \text{const}$ ) then the solution of the problem (19)–(21) satisfies the following a priori estimate:

$$\begin{aligned} \frac{1}{\Gamma(2-\alpha)} \sum_{j'=0}^j (t_{j-j'+1}^{1-\alpha} - t_{j-j'}^{1-\alpha}) \|y^{j'+1}\|_0^2 + c_1 \sum_{j'=0}^j \|\sigma y_{\bar{x}}^{j'+1} + (1-\sigma)y_{\bar{x}}^{j'}\|_0^2 \tau \leq \\ \leq \frac{l^2}{2c_1} \sum_{j'=0}^j \|\varphi^{j'}\|_0^2 \tau + \frac{t_{j+1}^{1-\alpha}}{\Gamma(2-\alpha)} \|u_0(x_i)\|_0^2. \end{aligned} \quad (31)$$

Here the results are obtained for the homogeneous boundary conditions  $u(0, t) = 0$ ,  $u(l, t) = 0$ . In the case of inhomogeneous boundary conditions  $u(0, t) = \mu_1(t)$ ,  $u(l, t) = \mu_2(t)$  the boundary conditions for the difference problem will have the following form:

$$y(0, t) = \mu_1(t), \quad y(l, t) = \mu_2(t). \quad (32)$$

Convergence of the difference scheme (19), (21), (32) follows from the results obtained above. Actually, let us introduce the notation  $y = z + u$ . Then the error  $z = y - u$  is a solution of the following problem:

$$\Delta_{0t_j}^{\alpha_i} z = \Lambda(\sigma z^{j+1} + (1-\sigma)z^j) + \psi, \quad i = 1, \dots, N-1, \quad j = 1, \dots, j_0-1, \quad (33)$$

$$z(0, t) = 0, \quad z(l, t) = 0, \quad j = 0, \dots, j_0, \quad (34)$$

$$z(x, 0) = 0, \quad i = 0, \dots, N, \quad (35)$$

where  $\psi \equiv \Lambda(\sigma u^{j+1} + (1-\sigma)u^j) - \Delta_{0t_j}^{\alpha_i} u + \varphi = O(\tau + h^2)$ .

The solution of the problem (33)–(35) satisfies the estimation (27) so that the solution of the difference scheme (19), (21), (32) converges to the solution of the corresponding differential problem with order  $O(\tau + h^2)$ .

### 3.2. Numerical results

In this section, the following variable order time fractional diffusion equation is considered:

$$\begin{cases} \partial_{0t}^{\alpha(x)} u = \frac{\partial}{\partial x} \left( k(x, t) \frac{\partial u}{\partial x} \right) - q(x, t)u + f(x, t), & 0 < x < 1, 0 < t \leq 1, \\ u(0, t) = \mu_1(t), \quad u(l, t) = \mu_2(t), & 0 \leq t \leq 1, \\ u(x, 0) = u_0(x), & 0 \leq x \leq l, \end{cases} \quad (36)$$

where  $\alpha(x) = \frac{5+4\sin(6x)}{10}$ ,  $k(x, t) = \frac{5+\cos(t)}{(3x^2+1)(3x^4+2x+1)+(x^3+x+1)(12x^3+2)}$ ,  $q(x, t) = \frac{1+\sin(t)}{(x^3+x+1)(3x^4+2x+1)}$ ,  $f(x, t) = (x^3+x+1)(3x^4+2x+1)\left(\frac{6t^{3-\alpha(x)}}{\Gamma(4-\alpha(x))} + \frac{6t^{2-\alpha(x)}}{\Gamma(3-\alpha(x))}\right) + (1+\sin(t))(t^3+3t^2+1)$ ,  $\mu_1(t) = t^3+3t^2+1$ ,  $\mu_2(t) = 18(t^3+3t^2+1)$ ,  $u_0(x) = (x^3+x+1)(3x^4+2x+1)$ .

The exact solution is  $u(x, t) = (x^3+x+1)(3x^4+2x+1)(t^3+3t^2+1)$ .

All the calculations are performed at  $\sigma = 1/(3-2^{1-\min(\alpha_i)})$ , where for the considered example  $\min(\alpha_i) = 1/10$ .

A comparison of the numerical solution and exact solution is provided in **Table 1**.

**Table 2** shows that if  $h = 1/500$ , then as the number of time of our approximate scheme is decreased, a reduction in the maximum error takes place, as expected and the convergence order of time is  $O(\tau)$ , where the convergence order is given by the following formula: Convergence order =  $\log_{\frac{\tau_1}{\tau_2}} \frac{e_1}{e_2}$ .

**Table 3** shows that when we take  $h^2 = \tau$ , as the number as spatial subintervals/time steps is decreased, a reduction in the maximum error takes place, as expected the convergence order of the approximate scheme is  $O(h^2 + \tau)$ , where the convergence order is given by the following formula: Convergence order =  $\log_{\frac{h_1}{h_2}} \frac{e_1}{e_2}$ .

**Table 1**The error, numerical solution and exact solution, when  $t = 1$ ,  $h = 1/10$ ,  $\tau = 1/100$ .

| Space( $x_i$ ) | Numerical solution | Exact solution | Error     |
|----------------|--------------------|----------------|-----------|
| 0.0000         | 5.0000000          | 5.0000000      | 0.0000000 |
| 0.1000         | 6.6068921          | 6.6076515      | 0.0007594 |
| 0.2000         | 8.4813103          | 8.4849920      | 0.0036817 |
| 0.3000         | 10.7677569         | 10.7772305     | 0.0094736 |
| 0.4000         | 13.7191945         | 13.7381760     | 0.0189815 |
| 0.5000         | 17.7407866         | 17.7734375     | 0.0326509 |
| 0.6000         | 23.4561082         | 23.5063040     | 0.0501958 |
| 0.7000         | 31.8041726         | 31.8738645     | 0.0696919 |
| 0.8000         | 44.1761384         | 44.2609280     | 0.0847896 |
| 0.9000         | 62.6016758         | 62.6793035     | 0.0776277 |
| 1.0000         | 90.0000000         | 90.0000000     | 0.0000000 |

**Table 2**Maximum error behavior versus time grid size reduction at  $t = 1$  when  $h = 1/500$ .

| $\tau$ | Maximum error | Convergence order |
|--------|---------------|-------------------|
| 1/256  | 0.0344960     |                   |
| 1/1024 | 0.0086690     | 0.996             |
| 1/4096 | 0.0021738     | 0.998             |

**Table 3**Maximum error behavior versus grid size reduction at  $t = 1$  when  $h^2 = \tau$ .

| $h$   | Maximum error | Convergence order |
|-------|---------------|-------------------|
| 1/40  | 0.0056275     |                   |
| 1/80  | 0.0014141     | 1.993             |
| 1/160 | 0.0003542     | 1.997             |

### 3.3. The Robin boundary value problem.

To the differential problem (1), (3), (12) we assign the following difference scheme:

$$\Delta_{0t_j}^{\alpha_i} y = \Lambda(\sigma y^{j+1} + (1 - \sigma)y^j) + \varphi, \quad i = 0, \dots, N, j = 1, \dots, j_0, \quad (37)$$

$$y(x, 0) = u_0(x), \quad i = 0, \dots, N, \quad (38)$$

where  $\Lambda y = (a_1 y_x - \tilde{\beta}_1 y)/(0.5h), i = 0, \quad \Lambda y = (a y_{\bar{x}})_x - dy, i = 1, \dots, N-1,$   
 $\Lambda y = (a_N y_{\bar{x}} - \tilde{\beta}_2 y)/(0.5h), i = N, \quad \varphi_0 = (2\tilde{\mu}_1)/h, \quad \varphi_N = (2\tilde{\mu}_2)/h, \quad \tilde{\beta}_1 =$   
 $\beta_1 + 0.5hd_0, \quad \tilde{\beta}_2 = \beta_2 + 0.5hd_N, \quad \tilde{\mu}_1 = \mu_1 + 0.5hf_0, \quad \tilde{\mu}_2 = \mu_2 + 0.5hf_N.$   
The difference scheme (37)–(38) has the order of approximation  $O(\tau + h^2)$ .

**Theorem 4.** The difference scheme (37)–(38) at  $\sigma \geq 1/(3 - 2^{1-\min(\alpha_i)})$  is absolutely stable and its solution satisfies the following a priori estimate:

$$\begin{aligned} & \left[ \frac{1}{\Gamma(2 - \alpha_i)}, \sum_{j'=0}^j (t_{j-j'+1}^{1-\alpha_i} - t_{j-j'}^{1-\alpha_i}) (y^{j'+1})^2 \right] + \\ & + \gamma \sum_{j'=0}^j \left( \|(y_{\bar{x}}^{(\sigma)})^{j'+1}\|_0^2 + ((y_0^{(\sigma)})^{j'+1})^2 + ((y_N^{(\sigma)})^{j'+1})^2 \right) \tau \leq \\ & \leq \frac{\delta}{\gamma} \sum_{j'=0}^j \left( (\tilde{\mu}_1^{j'+1/2})^2 + (\tilde{\mu}_2^{j'+1/2})^2 + \|\varphi^{j'}\|_0^2 \right) \tau + \\ & + \left[ \frac{t_{j+1}^{1-\alpha_i}}{\Gamma(2 - \alpha_i)}, u_0^2(x_i) \right], \end{aligned} \quad (39)$$

where  $\gamma = \min\{c_1, \beta_0\}$ ,  $\delta = \max\{1+l, l^2\}$ ,  $[y, v] = \sum_{i=1}^{N-1} y_i v_i h + 0.5y_0 v_0 h + 0.5y_N v_N h$ ,  $\|y\|_0^2 = [y, y]$ ,  $(y^{(\sigma)})^{j'+1} = \sigma y^{j'+1} + (1 - \sigma)y^{j'}$ .

**Proof.** Let us multiply scalarly equation (37) by  $y^{(\sigma)} = \sigma y^{j+1} + (1 - \sigma)y^j$ :

$$[\Delta_{0t_j}^{\alpha_i} y, y^{(\sigma)}] - [\Lambda y^{(\sigma)}, y^{(\sigma)}] = [\varphi, y^{(\sigma)}], \quad (40)$$

Let us transform the terms occurring in identity (40)

$$\begin{aligned} & [y^{(\sigma)}, \Delta_{0t_j}^{\alpha_i} y] \geq \frac{1}{2} [1, \Delta_{0t_j}^{\alpha_i} (y^2)] + \\ & + \left[ \frac{\tau^{\alpha_i} \Gamma(2 - \alpha_i)}{2(2 - 2^{1-\alpha_i})} ((3 - 2^{1-\alpha_i})\sigma - 1), (\Delta_{0t_j}^{\alpha_i} y)^2 \right], \\ & - [\Lambda y^{(\sigma)}, y^{(\sigma)}] = \tilde{\beta}_1 y_0^2 + \tilde{\beta}_2 y_N^2 + (a, (y_{\bar{x}}^{(\sigma)})^2) + [d, (y^{(\sigma)})^2], \\ & |[\varphi, y^{(\sigma)}]| \leq \varepsilon \|y^{(\sigma)}\|_0^2 + \tilde{\mu}_1 y_0 + \tilde{\mu}_2 y_N + \frac{1}{4\varepsilon} \|\varphi\|_0^2, \quad \varepsilon > 0. \end{aligned}$$

Taking into account the above performed transformations, from identity (40) at  $\sigma \geq 1/(3 - 2^{1-\min(\alpha_i)})$  one arrives at the inequality

$$\frac{1}{2} [1, \Delta_{0t_j}^{\alpha_i} (y^2)] + c_1 \|y_{\bar{x}}^{(\sigma)}\|_0^2 + \beta_0 (y_0^2 + y_N^2) \leq$$

$$\leq \varepsilon(\|y^{(\sigma)}\|_0^2 + y_0^2 + y_N^2) + \frac{1}{4\varepsilon}(\tilde{\mu}_1^2 + \tilde{\mu}_2^2 + \|\varphi\|_0^2). \quad (41)$$

From (40) at  $\varepsilon = \gamma/(2\delta)$ , using that  $\|y\|_0^2 \leq l^2\|y_{\bar{x}}\|_0^2 + l(y_0^2 + y_N^2)$ , one has the following inequality:

$$\begin{aligned} & [1, \Delta_{0t_j}^{\alpha_i}(y^2)] + \gamma(\|y_{\bar{x}}^{(\sigma)}\|_0^2 + y_0^2 + y_N^2) \leq \\ & \leq \frac{\delta}{\gamma}(\tilde{\mu}_1^2 + \tilde{\mu}_2^2 + \|\varphi\|_0^2). \end{aligned} \quad (42)$$

Multiplying inequality (42) by  $\tau$  and summing over  $j'$  from 0 to  $j$ , one obtains a priori estimate (39). The stability and convergence of the difference scheme (37)–(38) follow from the a priori estimate (39).

If  $\alpha(x) = \alpha$  ( $\alpha = \text{const}$ ), then for the solution of the problem (37)–(38) one has the following a priori estimate:

$$\begin{aligned} & \frac{1}{\Gamma(2-\alpha)} \sum_{j'=0}^j (t_{j-j'+1}^{1-\alpha} - t_{j-j'}^{1-\alpha}) \|y^{j'+1}\|_0^2 + \\ & + \gamma \sum_{j'=0}^j \left( \| (y_{\bar{x}}^{(\sigma)})^{j'+1} \|_0^2 + ((y_0^{(\sigma)})^{j'+1})^2 + ((y_N^{(\sigma)})^{j'+1})^2 \right) \tau \leq \\ & \leq \frac{\delta}{\gamma} \sum_{j'=0}^j \left( (\tilde{\mu}_1^{j'+1/2})^2 + (\tilde{\mu}_2^{j'+1/2})^2 + \|\varphi^{j'}\|_0^2 \right) \tau + \\ & + \frac{t_{j+1}^{1-\alpha}}{\Gamma(2-\alpha)} \|u_0(x_i)\|_0^2. \end{aligned} \quad (43)$$

#### 4. Conclusion

The results obtained in the present paper allow to apply the method of energy inequalities to finding a priori estimates for boundary value problems for the fractional diffusion equation in differential and difference settings exactly as in the classical case ( $\alpha(x) = 1$ ). It is interesting to note that the condition  $\sigma \geq 1/(3 - 2^{1-\alpha_i})$  at  $\alpha(x) = 1$  turns into the well known condition  $\sigma \geq 1/2$  of the absolute stability of the difference schemes for the classical diffusion equation.



## 5. Acknowledgements

Dedicated to Prof. M. Kh. Shkhanukov, on the occasion of his 75-th birthday.

This work was supported by the Russian Foundation for Basic Research (project 10-05-01150-a) and presented at the 4-th IFAC Workshop on Fractional Differentiation and Its Applications, Badajoz, Spain, October 18-20, 2010.

## References

- [1] A.M. Nahushev, Fractional Calculus and its Application, FIZMATLIT, Moscow, 2003 (in Russian).
- [2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
- [3] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
- [4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equation, Elsevier, Amsterdam, 2006.
- [5] V.V. Uchaikin, Method of Fractional Derivatives, Artishok, Uljanovsk, 2008 (in Russian).
- [6] T. M. Atanackovic, S. Pilipovic, Hamilton's principle with variable order fractional derivatives, *Fract. Calc. Appl. Anal.* 14(1) (2011) 94–109.
- [7] C.F.M. Coimbra, Mechanics with variable-order differential operators, *Ann. Phys.* 12 (1112) (2003) 692-703.
- [8] C.F. Lorenzo, T.T. Hartley, Variable order and distributed order fractional operators, *Nonlinear Dynam.* 29 (2002) 57-98.
- [9] S. Shen, F. Liu, J. Chen, I. Turner, V. Anh, Numerical techniques for the variable order time fractional diffusion equation, *Appl. Math. Comp.* 218 (2012) 10861–10870.

- [10] Chang-Ming Chen, F. Liu, V. Anh, I. Turner, Numerical methods for solving a two-dimensional variable-order anomalous subdiffusion equation, *Math. Comp.* 81 (2012) 345–366.
- [11] C. Chen, F. Liu, V. Anh, I. Turner, Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equations, *SIAM J. Scien. Comput.* 32(4) (2010) 1740–1760.
- [12] P. Zhuang, F. Liu, V. Anh, I. Turner, Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term, *SIAM J. Numer. Anal.* 47(3) (2009) 1760–1781.
- [13] R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of an explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, *Appl. Math. Comput.* 212 (2009) 435–445.
- [14] I. Podlubny, Matrix approach to discrete fractional calculus, *Fract. Calc. Appl. Anal.* 3(4) (2000) 359–386.
- [15] M. Kh. Shkhanukov-Lafishev, F.I. Taukenova, Difference methods for solving boundary value problems for fractional differential equations, *Comput. Math. Math. Phys.* 46(10) (2006) 1785–1795.
- [16] A. N. Kochubey, Diffusion of the fractional order, *Differentsialnye Uravneniya*. 26(4) (1990) 660–770.
- [17] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, *Appl. Math. Lett.* 9(6) (1996) 23–28.
- [18] F. Mainardi, R. Gorenflo, Time-fractional derivatives in relaxation processes: a tutorial survey, *Fract. Calc. Appl. Anal.* 10(3) (2007) 269–308.
- [19] A.V. Pskhu, The fundamental solution of a diffusion-wave equation of fractional order, *Izvestiya: Mathematics*, 73(2) (2009) 351–392 (in Russian).
- [20] M. Kh. Shkhanukov-Lafishev, About convergence of difference schemes for differential equations with fractional derivatives, *Doklady Akademii Nauk* 348(6) (1996) 746–748 (in Russian).

- [21] M. Kh. Shkhanukov-Lafishev, M.M. Lafisheva, Locally one-dimensional difference schemes for the fractional order diffusion equation, *Computational Mathematics and Mathematical Physics* 48(10) (2009) 1875–1884.
- [22] A.A. Alikhanov, A Priori Estimates for Solutions of Boundary Value Problems for Fractional-Order Equations, *Differ. Equ.* 46(5) (2010) 660–666.
- [23] A.V. Pskhu, *Partial Differential Equations of the Fractional Order*, Nauka, Moscow, 2005.
- [24] M. Caputo, *Elasticita e Dissipazione*, Zanichelli, Bologna, 1969.
- [25] Y. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation, *Comput. Math. Applic.* 59 (2010) 1766–1772.
- [26] Y. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, *J. Math. Anal. Appl.* 374 (2011) 538–548.
- [27] M. Meerschaert, E. Nane, P. Vellaisamy, Fractional Cauchy problems on bounded domains, *Ann. Probab.* 37 (2009) 979–1007.
- [28] A.A. Samarskiy, *Theory of Difference Schemes*, Nauka, Moscow, 1977.